that the integral shall exist with v in place of u. For since $u = v + j$ every sum S for u is the sum of two similar ones for v and j, and when f and u have no common discontinuity the last has a definite limit as δ approaches zero, as has been shown. This condition is, however, equivalent to that of the theorem. For the total variation of u on an interval is the sum of the total variations of v and j, and the fact that the set of points common to two denumerable sets of intervals is itself representable as such a denumerable set, makes it possible to show that the total variation of u on a set of points D is also the sum of the similar variations for v and j. Furthermore the total variation $J(x)$ of j is the value obtained from the series (5) by replacing each term by its absolute value, and on a set of points D containing no discontinuity of u it is zero. For if the ξ_k's are a set of points as described above, a denumerable set A of non-overlapping intervals can be selected approaching each ξ_k as a limit on both sides and containing all the points of ab except the ξ_k's. This set of intervals will also enclose D in its interior since D contains no ξ_k, and the sum of the total variations of j on A will surely be less than ϵ. It is easy to see, conversely, that when the variation of j is zero on D then the latter contains no discontinuity of u. Hence the existence of the integral with v in place of u and the conditions that f and u have no discontinuity in common imply that the total variation of u on D is zero, and conversely, which was to be proved.

TRANSFORMATIONS OF APPLICABLE CONJUGATE NETS OF CURVES ON SURFACES

By Luther Pfahler Eisenhart

DEPARTMENT OF MATHEMATICS, PRINCETON UNIVERSITY

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When the rectangular point coördinates x, y, z of a surface satisfy an equation of the form

$$\frac{\partial^2\theta}{\partial u \partial v} = a\frac{\partial\theta}{\partial u} + b\frac{\partial\theta}{\partial v}, \tag{1}$$

the curves u=const., v=const. form a conjugate system. We assume that the parametric system of curves is of this sort throughout this note, and we shall speak of it as a *net*. Equation (1) is the *point equation* of the net.

If N is such a net, the coördinates x', y', z' of a second net N' are given by quadratures of the form

$$\frac{\partial x'}{\partial u} = h\frac{\partial x}{\partial u}, \quad \frac{\partial x'}{\partial v} = l\frac{\partial x}{\partial v}, \tag{2}$$

provided that h and l are functions of u and v subject to the conditions

$$\frac{\partial h}{\partial v} = a(l-h), \quad \frac{\partial l}{\partial u} = b(h-l). \tag{3}$$

Moreover, each pair of solutions h, l of these equations leads by quadratures to a net N', such that the tangents at corresponding points M and M' to the curves of the net are parallel. All nets *parallel* to N are obtained in this way.

If θ is a solution of equation (1), and θ' is the corresponding function given by

$$\frac{\partial \theta'}{\partial u} = h\frac{\partial \theta}{\partial u}, \quad \frac{\partial \theta'}{\partial v} = l\frac{\partial \theta}{\partial v}, \tag{4}$$

then the functions x_1, y_1, z_1 defined by equations of the form

$$x_1 = x - \frac{\theta}{\theta'}x' \tag{5}$$

are the coördinates of a net N_1, so related to N that the lines joining corresponding points M and M_1 of these nets form a congruence whose developables meet the surfaces on which these nets lie in the curves of the nets. We say that the nets so related geometrically are in the relation of a *transformation* T. Parallel nets are in such relation. We have shown[1] that any transformation T of N into a non-parallel net N_1 is given by equations of the form (5). Hence any transformation T of N is determined by a parallel net and by a solution of the point equation of the net.

When two surfaces are applicable to one another, in the correspondence thus established there is a unique conjugate system of curves on one surface corresponding to a conjugate system on the other. We say that these nets are *applicable*. This paper is concerned with the transformations of applicable nets into applicable nets.

If we have two applicable nets N and $\bar{N}$ with the respective point coördinates x, y, z and $\bar{x}, \bar{y}, \bar{z}$, the analytical condition of their applicability is

$$\Sigma\left(\frac{\partial x}{\partial u}\right)^2 = \Sigma\left(\frac{\partial \bar{x}}{\partial u}\right)^2, \quad \Sigma\frac{\partial x}{\partial u}\frac{\partial x}{\partial v} = \Sigma\frac{\partial \bar{x}}{\partial u}\frac{\partial \bar{x}}{\partial v}, \quad \Sigma\left(\frac{\partial x}{\partial v}\right)^2 = \Sigma\left(\frac{\partial \bar{x}}{\partial v}\right)^2 \tag{6}$$

the sign Σ indicating the summation of terms in x, y and z. Since the coefficients a and b of equation (1) are functions of the left-hand members of (6) and their derivatives, equation (1) is likewise the point equation of the net $\overline{N}$. In view of this fact a pair of functions h and l satisfying (3) leads to a net $\overline{N}'$ parallel to $\overline{N}$ as well as to N' parallel to $\overline{N}$. Moreover, the nets N' and $\overline{N}'$ are applicable. This result is due to Peterson.[2] The common point equation of the nets N' and $\overline{N}'$ admits the solution

$$\theta' = x'^2 + y'^2 + z'^2 - \bar{x}'^2 - \bar{y}'^2 - \bar{z}'^2. \tag{7}$$

This function and the corresponding function θ given by the quadrature (4) determine transforms N_1 and $\overline{N}_1$ of N and $\overline{N}$ respectively, and these new nets are applicable. Hence each net parallel to one of two applicable nets determines a parallel to the other, and by a further quadrature two applicable nets which are T transforms of the original nets. Moreover, the function θ' given by (7) is the only one leading to such a result. This result can be generalized at once to applicable nets in space of any order.

Nets are of three types with regard to applicability. Nets of the first type do not admit any applicable nets. Those of the second type admit one applicable net, whereas each net of the third type admits an infinity of applicable nets. We say that the latter are permanent in deformation, and for the sake of brevity call them *permanent nets.* Every net parallel to a permanent net is a permanent net, and each of the infinity of nets applicable to the one is parallel to one of the infinity applicable to the other by the method of Peterson. Suppose now that we have a permanent net N, two applicable nets $\overline{N}$ and $\overline{\overline{N}}$, and the respective parallel applicable nets N', $\overline{N}'$, $\overline{\overline{N}}'$. By the process of the preceding paragraph we obtain two transforms N_1 and N_2 of N, in general distinct, such that corresponding points of N, N_1 and N_2 lie on the same line, whose direction-parameters are the coördinates of N'. At the same time we obtain two transforms of $\overline{N}$ and two of $\overline{\overline{N}}$. As N admits an infinity of applicable nets, this process can be extended with the result that, in general, N and each of its deforms admit an infinity of transforms. We have raised the question whether in any case this infinity of transforms were coincident for each of the nets so that we obtain a permanent net N_1, whose infinity of applicable nets are the T transforms of the nets applicable to N. We refer to this question as *Problem A*.

Permanent nets belong to the general class of nets whose tangential coördinates satisfy an equation of Laplace with equal invariants. We have established[3] the existence of transformations T of nets of this kind into similar nets. When in particular the given N is a permanent net, by the solution of a completely integrable system of partial differential equations of the first order a family of parallel nets of a particular type are obtained, each of which determines a T transform N_1, which also is a permanent net. All of these transformations are now shown to give a solution of Problem A.

In the transformations just referred to we did not consider permanent nets for which the curves in one family are represented on the Gauss sphere by one system of the imaginary generators. Drach[4] solved the problem of the deformation of nets of this kind. We show how in two ways these nets can be transformed into nets of the same kind as a solution of Problem A.

The third type of permanent nets are those whose two families of curves are represented on the sphere by its isotropic generators. These curves are the minimal lines on a minimal surface. There are no transformations of nets of this kind into similar nets furnishing a solution of Problem A.

[1] Eisenhart, *Trans. Amer. Math. Soc., New York*, **18**, 1917, (97-124).
[2] Peterson, *Ueber Curven und Flachen*, Moskau and Leipzig, 1868, (106).
[3] Eisenhart, *Rend. Circ. Mat., Palermo*, **39**, 1915, (153-176).
[4] Drach, *Ann. Fac. Sci. Toulouse*, (Ser. 2), **10**, 1908, (125-164).

ON BILINEAR AND N-LINEAR FUNCTIONALS

By Charles Albert Fischer

DEPARTMENT OF MATHEMATICS, COLUMBIA UNIVERSITY

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It has been proved by Riesz[1,2] that if a linear functional $A(f(s))$ is continuous with zeroth order, there is a unique regular function $\alpha(s)$ which satisfies the equation

$$A(f) = \int f(s)\, d\alpha(s),$$

and the variation of α is the least upper bound of the expression $|A(f)|/mf$, where mf is the maximum of $|f(s)|$. From this theorem Fréchet][3] has proved that if $U(f(s), g(t))$ is bilinear, that is linear in each argument, there is a function $u(s, t)$ which is regular in t and satisfies the equation

$$U(f, g) = \int\int f(s)\, g(t)\, d_2 u(s, t), \tag{1}$$

and by modifying the definition of the variation of a function of two variables, he has proved that the variation of $u(s,t)$ is the least upper bound of $|U(f, g)|/mfmg$.